1. Consider the planes $x + y + z = 1$ and $x - y + z = 2$.

(a) (5 points) Recall that the angle between two planes is by definition equal to the angle between their normals. If θ is the angle between the two given planes, find $\cos\theta$.

Normal vectors are $\vec{n_1} = (1,1,1)$, $\vec{n_2} = (1,-1,1)$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{1\cdot 1 + 1\cdot(-1) + 1\cdot 1}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+(-1)^2+1^2}} = \boxed{\frac{1}{3}}$$

Note The statement of the problem is slightly ambiguous, as you can also get $\cos\theta = -\frac{1}{3}$ by considering $-\vec{n_1}$ instead of $\vec{n_1}$.

(b) (5 points) Find a plane that is orthogonal to the two given planes and which goes through the point $(2, 1, 3)$.

A normal vector $\vec{n}$ should be orthogonal to both $\vec{n_1}$ and $\vec{n_2}$

$$\Rightarrow \vec{n} = \vec{n_1} \times \vec{n_2} = (1,1,1)\times(1,-1,1) = (2,0,-2)$$

The plane equation is given by

$$\boxed{2(x-2) + 0\cdot(y-1) - 2(z-3) = 0}$$

Note The answer can be given in many other forms, such as $x - z + 1 = 0$.

2. (10 points) Consider $f(x,y) = x^2 + \frac{y^2}{4}$. Let P be the point with $(x,y) = (1,2)$. Find the gradient ∇f, the direction (unit vector) of fastest increase, the direction (unit vector) of fastest decrease, and the direction in which the function neither increases nor decreases at the point P ($(x,y) = (1,2)$).

$$\nabla f = (f_x, f_y) = \left(2x, \frac{y}{2}\right) \Rightarrow \nabla f(1,2) = \boxed{(2,1)}$$

The direction of fastest increase is given by $\nabla f(1,2)$.

Its unit vector is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{(2,1)}{\sqrt{2^2+1^2}} = \boxed{\frac{1}{\sqrt{5}}(2,1)}$$

The direction of fastest decrease is given by $-\nabla f(1,2)$.

Its unit vector is

$$-\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = -\frac{(2,1)}{\sqrt{2^2+1^2}} = \boxed{-\frac{1}{\sqrt{5}}(2,1)}$$

Let $\vec{u} = (a,b)$ be the unit vector along which the directional derivative of f is zero at $P = (1,2)$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = (2,1)\cdot(a,b) = 2a+b$$

$$D_{\vec{u}} f(1,2) = 0 \Rightarrow 2a+b=0 \Rightarrow b = -2a$$

$$\vec{u} = \pm \frac{(1,-2)}{|(1,-2)|} = \pm \frac{(1,-2)}{\sqrt{1^2+(-2)^2}} = \boxed{\pm \frac{1}{\sqrt{5}}(1,-2)}$$

3. (10 points) Let $f(x, y) = F(x^2 + y^2) + G(xy)$, where F and G are functions of a single variable. Assume $F(2) = 1$, $F'(2) = 2$ and $G(-1) = -1$, $G'(-1) = -2$. Find $f(x, y)$, $\partial f/\partial x$ and $\partial f/\partial y$ at $(x, y) = (1, -1)$.

$$f(1,-1) = F(1^2+(-1)^2) + G(1\cdot(-1))$$

$$= F(2) + G(-1) = 1 - 1 = \boxed{0}$$

$$f_x = \frac{\partial}{\partial x}\left(F(x^2+y^2) + G(xy)\right) = \frac{\partial}{\partial x} F(x^2+y^2) + \frac{\partial}{\partial x} G(xy)$$

$$= F'(x^2+y^2)\cdot 2x + G'(xy)\cdot y$$

$$f_x(1,-1) = F'(1^2+(-1)^2)\cdot 2\cdot 1 + G'(1\cdot(-1))\cdot(-1)$$

$$= F'(2)\cdot 2\cdot 1 + G'(-1)\cdot(-1)$$

$$= 2\cdot 2 + (-2)\cdot(-1) = \boxed{6}$$

$$f_y = \frac{\partial}{\partial y}\left(F(x^2+y^2) + G(xy)\right) = \frac{\partial}{\partial y} F(x^2+y^2) + \frac{\partial}{\partial y} G(xy)$$

$$= F'(x^2+y^2)\cdot 2y + G'(xy)\cdot x$$

$$f_y(1,-1) = F'(1^2+(-1)^2)\cdot 2\cdot(-1) + G'(1\cdot(-1))\cdot 1$$

$$= F'(2)\cdot 2\cdot(-1) + G'(-1)\cdot 1$$

$$= 2\cdot 2\cdot(-1) + (-2) = \boxed{-6}$$

4. Consider the plane $x + 2y + 2z = 4$.

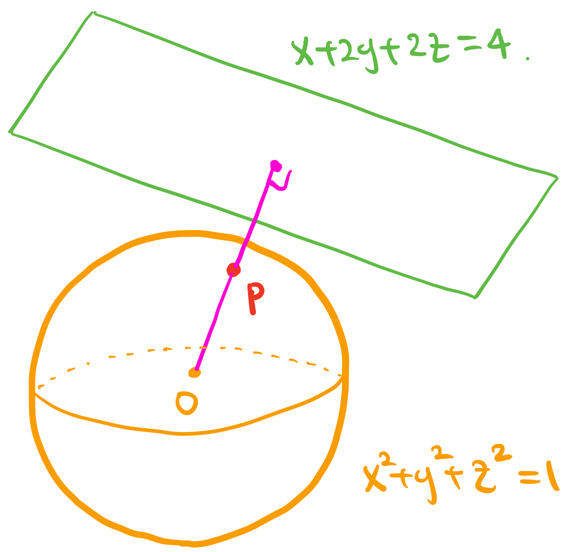
(a) (5 points) Find the distance from the plane to the origin.

The distance from the plane $x+2y+2z-4=0$ to the origin $(0,0,0)$ is equal to

$$\frac{|0+2\cdot 0+2\cdot 0-4|}{\sqrt{1^2+2^2+2^2}} = \boxed{\frac{4}{3}}$$

(b) (5 points) Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is closest to the plane.

Let P be the point on the sphere which is closest to the plane.



$\overrightarrow{OP}$ must be a normal vector of the plane $x+2y+2z=4$.

$\Rightarrow \overrightarrow{OP}$ is parallel to $\vec{n} = (1,2,2)$

Also $\overrightarrow{OP}$ represents a radius of the sphere $x^2+y^2+z^2=1$.

$\Rightarrow |\overrightarrow{OP}| = 1$.

Hence $\overrightarrow{OP}$ is the unit vector in the direction of $\vec{n}$

$$\Rightarrow \overrightarrow{OP} = \frac{\vec{n}}{|\vec{n}|} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}(1,2,2)$$

$$\Rightarrow P = \boxed{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)}$$

5. (10 points) The pressure P, volume V, and temperature T of one mole of an ideal gas satisfy $PV = RT$, where R is a constant. Suppose R is measured using $R = PV/T$.

(a) (5 points) Find the differential dR.

$$dR = \frac{\partial R}{\partial P} dP + \frac{\partial R}{\partial V} dV + \frac{\partial R}{\partial T} dT$$

$$= \boxed{\frac{V}{T} dP + \frac{P}{T} dV - \frac{PV}{T^2} dT}$$

(b) (5 points) If the percentage errors in the measurement of P, V, and T are 1%, 2%, and 3%, respectively, find the maximum percentage error in R.

The errors are represented by the differentials.

The error $dR = \frac{V}{T} dP + \frac{P}{T} dV - \frac{PV}{T^2} dT$ attains the maximum with $dP = 0.01P$, $dV = 0.02V$, $dT = -0.03T$.

$$\Rightarrow dR = \frac{V}{T} 0.01P + \frac{P}{T} 0.02V - \frac{PV}{T^2}(-0.03T)$$

$$= 0.01 \cdot \frac{PV}{T} + 0.02 \cdot \frac{PV}{T} + 0.03 \frac{PV}{T}$$

$$= 0.06 \cdot \frac{PV}{T} = 0.06R.$$

The maximum percentage error in R is

$$\frac{dR}{R} \cdot 100 = \frac{0.06R}{R} \cdot 100 = \boxed{6\ (\%)}$$

6. Consider the surface $xyz = 1$ in $\mathbb{R}^3$. Let P be the point $(x, y, z) = (2, 1, \frac{1}{2})$.

(a) (5 points) Find the normal to the surface at the point P.

The surface is a level surface of $f(x,y,z) = xyz$.

$\nabla f = (f_x, f_y, f_z) = (yz, zx, xy)$.

A normal vector at $P = (2, 1, \frac{1}{2})$ is given by

$\nabla f(2,1,2) = \boxed{(\frac{1}{2}, 1, 2)}$

(b) (5 points) Find the tangent plane to the surface at the point P.

The tangent plane to the surface at $P = (2, 1, \frac{1}{2})$ is

$$\boxed{\frac{1}{2}(x-2) + 1\cdot(y-1) + 2(z-\frac{1}{2}) = 0}$$

<u>Note</u> You can solve this problem using the method from section 14.4.

$xyz = 1 \rightsquigarrow z = \frac{1}{xy}$

$\Rightarrow$ The surface is the graph of $g(x,y) = \frac{1}{xy}$.

$g_x = -\frac{1}{x^2y} \Rightarrow g_x(2,1) = -\frac{1}{4}$.

$g_y = -\frac{1}{xy^2} \Rightarrow g_y(2,1) = -\frac{1}{2}$

The tangent plane at $P = (2, 1, \frac{1}{2})$ is given by

$z = g(2,1) + g_x(2,1)(x-2) + g_y(2,1)(y-1)$

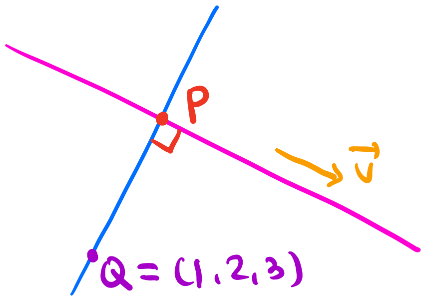
$$\Rightarrow \boxed{z = \frac{1}{2} - \frac{1}{4}(x-2) - \frac{1}{2}(y-1)}$$

7. An unknown line through the point $(1, 2, 3)$ intersects the given line $\mathbf{r}(t) = (t+3, 3-t, t+2)$ at right angles

(a) (7 points) Find the point of intersection of the two lines.

We find the value of t at the intersection point P.

The given line has a direction vector $\vec{v} = (1, -1, 1)$.



Set $Q = (1, 2, 3)$

$\Rightarrow \overrightarrow{PQ}$ is perpendicular to $\vec{v}$.

$\Rightarrow \overrightarrow{PQ} \cdot \vec{v} = 0$

$\Rightarrow (-t-2, t-1, -t+1) \cdot (1, -1, 1) = 0$

$\rightsquigarrow -3t = 0 \rightsquigarrow t = 0$.

$\Rightarrow P = \vec{r}(0) = \boxed{(3, 3, 2)}$

(b) (3 points) Find the parametric equation of the unknown line.

The unknown line passes through $P = (3, 3, 2)$ with a direction vector $\overrightarrow{PQ} = (-2, -1, 1)$.

$\Rightarrow$ The unknown line is parametrized by

$$\vec{\ell}(t) = \boxed{(3-2t,\ 3-t,\ 1+t)}$$

<u>Note</u> The answer can be given in many other forms, such as $\vec{\ell}(t) = (1+2t, 2+t, 3-t)$.

8. In this problem, take $g = 10$ m/s^2. A projectile is fired at an angle α, $0 < \alpha < 90^o$ on level ground at a speed of 120 m/s. Let $\mathbf{r}(t)$ be its position as a function of time.

(a) (3 points) Take $\mathbf{r}(0) = (0,0)$, $\mathbf{r}'(0) = 120(\cos\alpha, \sin\alpha)$, and $\mathbf{r}''(t) = (0, -g)$, and calculate $\mathbf{r}(t)$. Wind resistance is ignored.

$$\vec{r}\,'(t) = \vec{r}\,'(0) + \int_0^t \vec{r}\,''(u)\,du = 120(\cos\alpha, \sin\alpha) + \int_0^t (0,-10)\,du$$

$$= 120(\cos\alpha, \sin\alpha) + (0, -10t)$$

$$= (120\cos\alpha,\ 120\sin\alpha - 10t)$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(u)\,du$$

$$= (0,0) + \int_0^t (120\cos\alpha,\ 120\sin\alpha - 10u)\,du$$

$$= \boxed{(120\cos\alpha\, t,\ 120\sin\alpha\, t - 5t^2)}$$

(b) (3 points) Express the range of the projectile as a function of α.

The range is given by the x-coordinate when the projectile hits the ground.

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$$y=0 \Rightarrow 120\sin\alpha\, t - 5t^2 = 0 \Rightarrow t = \cancel{0},\ 24\sin\alpha$$

$$\Rightarrow \text{The range is } 120\cos\alpha \cdot 24\sin\alpha = \boxed{2880\sin\alpha\cos\alpha}$$

(c) (4 points) If the range is 720 m, find the two possible values of α. You may use the trig identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

$$720 = 2880\sin\alpha\cos\alpha = 1440\sin(2\alpha)$$

$$\Rightarrow \sin(2\alpha) = \frac{1}{2} \Rightarrow 2\alpha = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \alpha = \boxed{\frac{\pi}{12}, \frac{5\pi}{12}}$$